Additively Damped AFAC Variants:
Solver Construction

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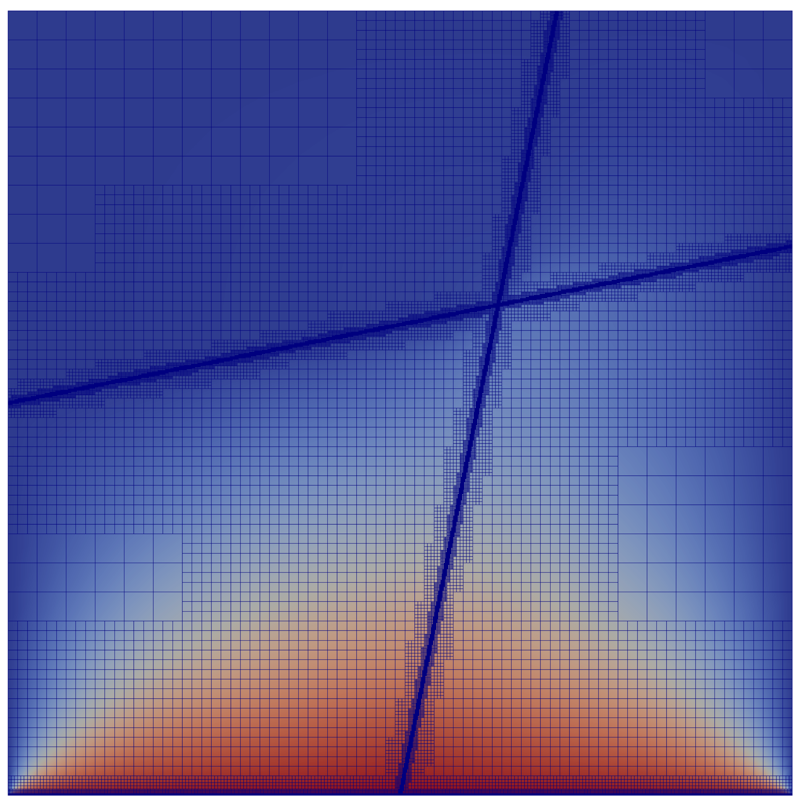
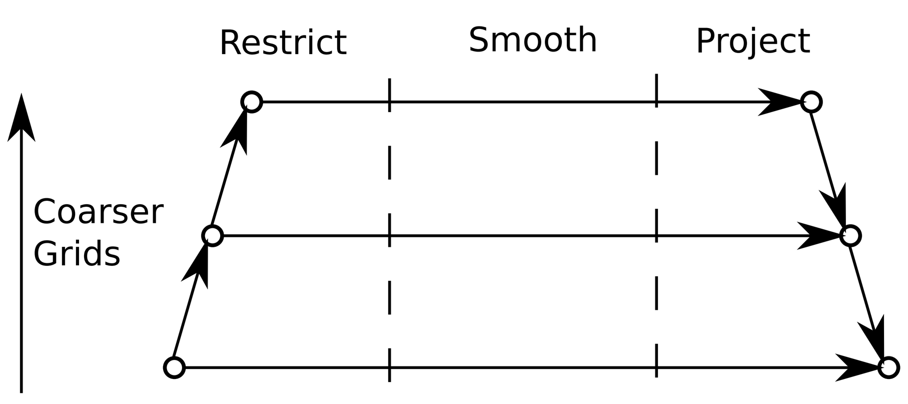
Introduction



- Solve the variable coefficient Poisson equation $-\nabla(\epsilon \cdot \nabla) u = f$
- Discontinuous material parameter ϵ
- Complicated ϵ means expensive assembly and tricky coarse grid corrections

Additive Multigrid

- Additive Multigrid removes sequential smoothing between levels
- Smooth on multiple levels—no synchronisation performed
- Increases potential for parallelism
- Reduces stability due to overshooting



V-cycle for Additive Multigrid

Stabilising Additive Multigrid

- Motivated by two level $V(1,0)$ Multigrid cycle:

$$\begin{aligned} u_{\ell,\text{mult}} \leftarrow\ & PA_{\ell-1}^{-1}R(b_\ell - A_\ell u_\ell) \\ & -PA_{\ell-1}^{-1}RA_\ell\omega_\ell M_\ell^{-1}(b_\ell - A_\ell u_\ell) \\ & +\left[u_\ell + \omega_\ell M_\ell^{-1}(b_\ell - A_\ell u_\ell)\right] \end{aligned}$$

- Compared to two level Additive cycle:

$$\begin{aligned} u_{\ell,\text{add}} \leftarrow\ & PA_{\ell-1}^{-1}R(b_\ell - A_\ell u_\ell) \\ & +\left[u_\ell + \omega_\ell M_\ell^{-1}(b_\ell - A_\ell u_\ell)\right] \end{aligned}$$

- Determine difference:

$$-PA_{\ell-1}^{-1}RA_\ell\omega_\ell M_\ell^{-1}(b_\ell - A_\ell u_\ell)$$

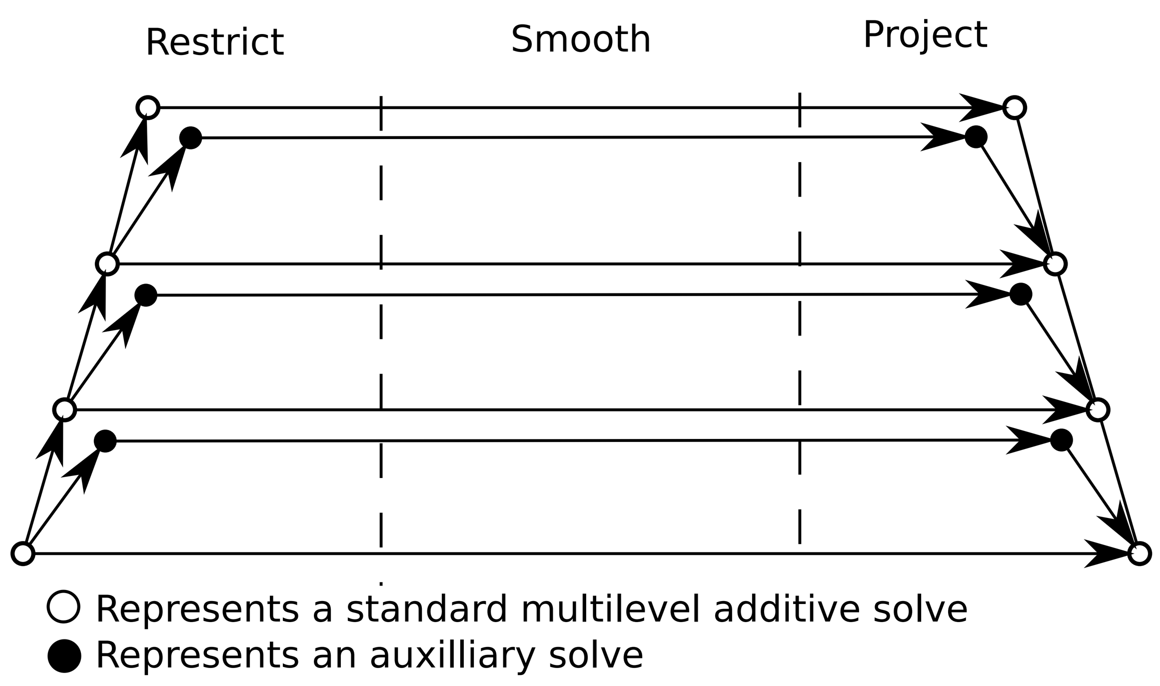
- Use additional additive damping term:

$$-PM_{\ell-1}^{-1}RA_\ell\omega_\ell M_\ell^{-1}(b_\ell - A_\ell u_\ell)$$

- Additional smoothing step built into the restriction operator, i.e. smoothed restriction operator $\tilde{R} = RA_\ell M_\ell^{-1}$

Auxilliary grids

- Inspired by AFAC (Asynchronous Fast Adaptive Composite grids)
- Introduce additional "auxilliary" grids
- Geometric multigrid—specifically space trees
- Locally Cartesian but allow dynamic AMR
- Dictates sparsity patterns—FE stencils have local support
- Auxilliary grids compute damping parameters for original grids so all grid levels handled in parallel

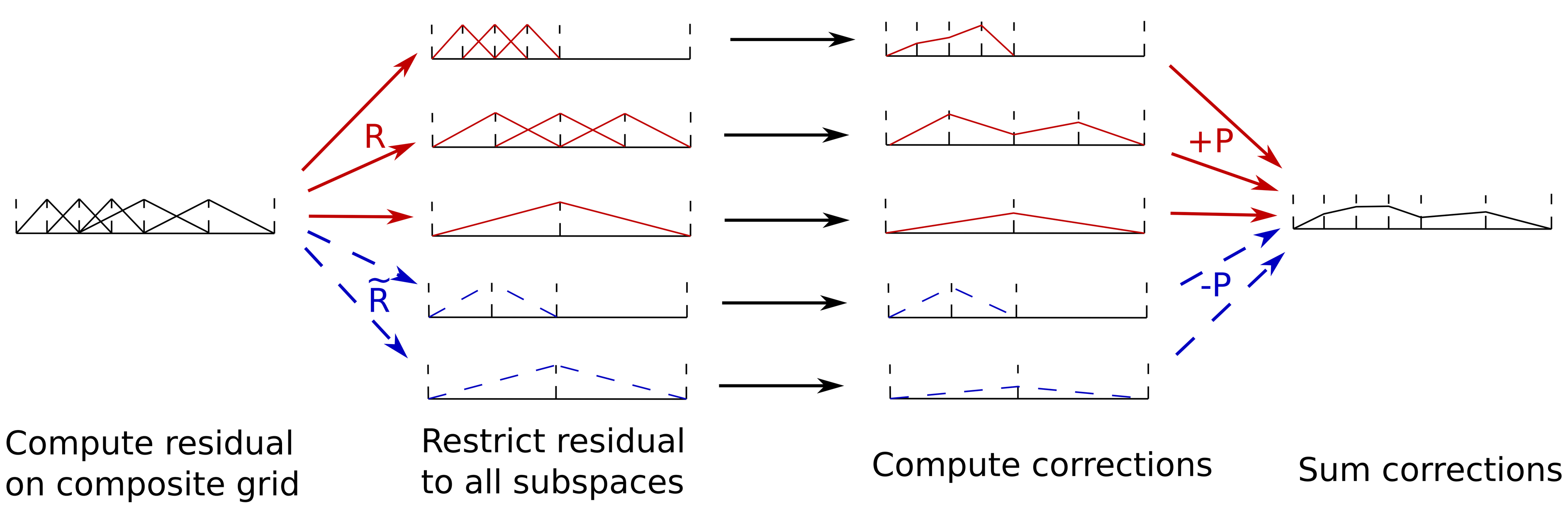


Modified V-cycle for our adAFAC-Jac method

- We remove sequential component between smooths i.e. damping parameter constructed in parallel with all other smoothing steps

Implementation

- Call adaFAC-Jac - an **a**dditively **d**amped AFAC with an additional **Jac**obi smoothing step



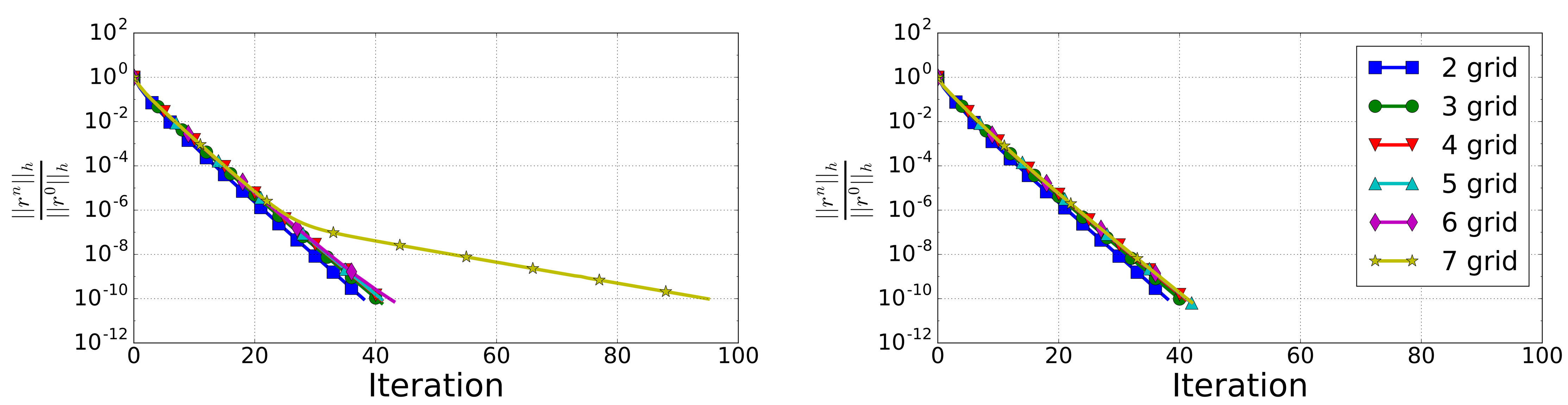
Data flow adaFAC-Jac as it restricts to different basis spaces to smooth and produce corrections

Algorithm 1 Blueprint of our algorithm and damping parameter computation

Input b_f fine grid right hand side
Parameter $R^{(i-j)}$ restriction to level i to level j and $P^{(i-j)}$ projection from level j to level i
Output New solution approximate $u_{\ell_{\text{max}}}$

function ADAFAC-JAC
  $r_{\ell_{\text{max}}} \leftarrow b_f - A_{\ell_{\text{max}}} u_{\ell_{\text{max}}}$
  for all $\ell_{\text{min}} \leq \ell \leq \ell_{\text{max}}$ **do**
    $b_\ell \leftarrow R^{\ell_{\text{max}}-\ell} r_{\ell_{\text{max}}}$ ▷ Restrict fine grid residual to grid level ℓ
  end for
  for all $\ell_{\text{min}} \leq \ell < \ell_{\text{max}}$ **do**
    $\tilde{b}_\ell \leftarrow \tilde{R}^{\ell_{\text{max}}-\ell} r_{\ell_{\text{max}}}$ ▷ Additional restriction residual into additional grid space
  end for
  for all $\ell_{\text{min}} < \ell \leq \ell_{\text{max}}$ **do**
    $c_\ell \leftarrow 0; \tilde{c}_\ell \leftarrow 0$ ▷ Initial "guess" for correction and damping parameter
    JACOBI($A_\ell c_\ell = b_\ell, \omega$) ▷ Iterate of correction equation stored in c_ℓ
    JACOBI($A_{\ell-1}\tilde{c}_{\ell-1} = \tilde{b}_{\ell-1}, \tilde{\omega}$) ▷ Iterate of damping equation (one level coarser)
  end for
  $c_{\ell_{\text{min}}} \leftarrow 0$
  JACOBI($A_{\ell_{\text{min}}} c_{\ell_{\text{min}}} = b_{\ell_{\text{min}}}, \omega$) ▷ Coarsest grid update (no additional damping performed)
  $u_{\ell_{\text{max}}} \leftarrow u_{\ell_{\text{max}}} + c_{\ell_{\text{min}}} + \sum_{\ell=\ell_{\text{min}-1}}^{\ell_{\text{max}}} P^{\ell_{\text{max}}-\ell} c_\ell - P^{\ell_{\text{max}}-(\ell-1)} \tilde{c}_{\ell-1}$
end function

Stability Comparison



Additive Multigrid (left); adaFAC-Jac (right)